

Universal Subleading Spectrum of Effective String Theory

J. M. Drummond

Department of Mathematics, Trinity College Dublin

E-mail: jmd@maths.tcd.ie

ABSTRACT: We analyse the spectrum of the D-dimensional Poincaré invariant effective string model of Polchinski and Strominger. It is shown that the leading terms beyond the Casimir term in the long distance expansion of the spectrum have a universal character which follows from the constraint of Poincaré invariance.

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1. Introduction

The question of whether QCD can be described by a string theory is one which has commanded much attention over several decades but is still unresolved. It is widely believed that the colour flux between a quark-antiquark pair can be described as a string for sufficiently large separations. The question of which, if any, string theory is relevant for such systems is a difficult problem. In [1] Polchinski and Strominger showed that it was possible to construct a string theory with manifest D-dimensional Poincaré symmetry. This string theory evades the traditional restriction to the critical dimension of 26 by including a term in the action which is valid only for expansion around a ‘long string’ vacuum, i.e. it diverges if the string is allowed to shrink to zero size. The coefficient of the new term can then be adjusted to cancel the anomalous central charge outside 26 dimensions.

We discuss the effective string model of [1] and show that the constraints of D-dimensional Poincaré invariance imply a universal subleading behaviour in the spectrum. Specifically we show that the R^{-3} terms are fixed and have the same form as in the Nambu-Goto spectrum. Such higher order terms in the spectrum are relevant for comparisons to lattice simulations of QCD flux tubes and other string-like solitons [2, 3, 4, 5].

2. Effective String Model

In this section we very briefly reexamine the D-dimensional covariant string model [1] and we refer the reader to the original article for further detail. The effective string action given in [1] is

$$S = \frac{1}{4\pi} \int d\tau^+ d\tau^- \left[\frac{1}{a^2} \partial_+ X \cdot \partial_- X + \beta \frac{\partial_+^2 X \cdot \partial_- X \partial_+ X \cdot \partial_-^2 X}{(\partial_+ X \cdot \partial_- X)^2} \right]. \quad (2.1)$$

As noted in [1] the non-polynomial terms in the action are no problem as long as one only considers expanding the theory about a ‘long string’ vacuum where the first derivatives $\partial_+ X$ and $\partial_- X$ are never small. Accordingly we consider the string to be wrapped around a spatial dimension compactified on a circle of radius R ¹. The ground state is given by

$$X_{\text{cl}}^\mu = e_+^\mu R\tau^+ + e_-^\mu R\tau^-. \quad (2.2)$$

¹Such an object is referred to as a ‘torelon’ in the QCD flux tube context.

The first derivatives of X are then of order R (which we take to be large) and higher derivatives are order one.

Adopting the notation that $Z = \partial_+ X \cdot \partial_- X$, we write this as

$$S = \frac{1}{4\pi} \int d\tau^+ d\tau^- \left[\frac{1}{a^2} Z + \beta \frac{\partial_+^2 X \cdot \partial_- X \partial_+ X \cdot \partial_-^2 X}{Z^2} \right]. \quad (2.3)$$

We will call the two terms S_0 and S_β respectively.

The new term in the action was motivated by the fact that in a properly covariant treatment of the underlying field theory one would expect some determinants to appear on the transformation from the field theory variables to the string theory variables. The new term is then precisely the Polyakov determinant in terms of the induced metric instead of an intrinsic metric. One can check that the term S_β is the only term up to $O(R^{-2})$ which obeys the requirements that it must have inverse powers only of the operator $Z = \partial_+ X \cdot \partial_- X$ (which has a large classical expectation value) and is not proportional to the lowest order equations of motion $\partial_+ \partial_- X = 0$ (otherwise it would be removable by a field redefinition) or the lowest order energy-momentum tensor $\partial_- X \cdot \partial_- X$ or $\partial_+ X \cdot \partial_+ X$ (which vanishes between physical states).

It is worth noting that one can rewrite the term S_β in a slightly simpler form. Using integration by parts ² one finds

$$S = \frac{1}{4\pi} \int d\tau^+ d\tau^- \left[\frac{1}{a^2} Z + \beta \frac{\partial_+^2 X \cdot \partial_-^2 X}{Z} \right]. \quad (2.4)$$

The modified conformal transformation law,

$$\delta X = \epsilon^-(\tau^-) \partial_- X - \frac{\beta a^2}{2} \partial_-^2 \epsilon^-(\tau^-) \frac{\partial_+ X}{Z} + (+ \longleftrightarrow -), \quad (2.5)$$

can be written as $\delta X = \delta_0 X + \delta_\beta X$ and is such that

$$\delta_0 S_0 = 0, \quad \delta_0 S_\beta + \delta_\beta S_0 = 0. \quad (2.6)$$

These equations holds exactly, i.e. to all orders in R^{-1} , a point not stressed in the original paper [1]. The non vanishing term $\delta_\beta S_\beta$ must be cancelled by appropriate extra terms, S_{β^2} in the Lagrangian and $\delta_{\beta^2} X$ in the transformation law. Thus the perturbation S_β generates an infinite series of extra terms to make the Lagrangian invariant. The continuation of the invariant at higher orders is not necessarily unique and the presence of free parameters signals the start of a new invariant.

The term $\delta_\beta S_\beta$ is $O(R^{-4})$ in the expansion of inverse string length. This implies that the required corrections to the action and transformation law only produce variations at this order and therefore corrections to the energy momentum tensor at $O(R^{-3})$. Any other new term S_γ in the action must obey the same constraints which were applied in the construction of S_β , namely that it must have inverse powers only of the operator

²We have no boundary and although we wrap the string around a compact dimension, the fields $\partial_+ X$ and $\partial_- X$ are still periodic.

$Z = \partial_+ X \cdot \partial_- X$ and that it must not be proportional the lowest order equations of motion or the lowest order energy momentum tensor. The first new terms which satisfy these requirements are in fact of order R^{-6} ³ and a basis which makes this explicit is:

$$L_1 = \frac{1}{Z^3} \partial_-^2 X \cdot \partial_-^2 X \partial_+^2 X \cdot \partial_+^2 X, \quad (2.7)$$

$$L_2 = \frac{1}{Z^3} \partial_-^2 X \cdot \partial_+^2 X \partial_-^2 X \cdot \partial_+^2 X, \quad (2.8)$$

$$L_3 = \frac{1}{Z^4} \partial_-^2 X \cdot \partial_+^2 X \partial_- X \cdot \partial_+^2 X \partial_-^2 X \cdot \partial_+ X, \quad (2.9)$$

$$L_4 = \frac{1}{Z^5} \partial_- X \cdot \partial_+^2 X \partial_- X \cdot \partial_+^2 X \partial_-^2 X \cdot \partial_+ X \partial_-^2 X \cdot \partial_+ X. \quad (2.10)$$

The coefficients of these terms will be constrained by the requirements of classical and quantum conformal invariance. Given all of the above we certainly expect the energy momentum tensor derived just from the the first corrections, S_β and $\delta_\beta X$, to be valid up to $O(R^{-2})$ and we will show that it is rather simple to deduce the physical spectrum up to $O(R^{-3})$ from these terms.

From the lowest order action and transformation law we find the standard energy-momentum tensor at lowest order,

$$T_{--}^0 = -\frac{1}{2a^2} \partial_- X \cdot \partial_- X, \quad (2.11)$$

The order β corrections to the action and transformation law give (after some calculation),

$$T_{--}^\beta = -\frac{\beta}{2} \left[-\frac{1}{Z} \partial_-^3 X \cdot \partial_+ X + \frac{1}{Z^2} (-\partial_- X \cdot \partial_- X \partial_-^2 X \cdot \partial_+^2 X + \partial_-^2 X \cdot \partial_- X \partial_- X \cdot \partial_+^2 X + \partial_-^2 X \cdot \partial_+ X \partial_-^2 X \cdot \partial_+ X) \right]. \quad (2.12)$$

To verify that this is conserved, one must calculate the equation of motion for X . Up to $O(R^{-3})$ this reads,

$$\partial_+ \partial_- X = \frac{\beta a^2}{2Z^2} (\partial_- X \partial_-^2 X \cdot \partial_+^3 X + \partial_+ X \partial_-^3 X \cdot \partial_+^2 X - \partial_-^2 X \partial_- X \cdot \partial_+^3 X - \partial_+^2 X \partial_-^3 X \cdot \partial_+ X) + O(R^{-4}). \quad (2.13)$$

Employing this one finds that $\partial_+ T_{--} = O(R^{-3})$. We now expand this in terms of the fluctuation field, Y , which is defined by

$$X^\mu = e_+^\mu R \tau^+ + e_-^\mu R \tau^- + Y^\mu. \quad (2.14)$$

The lowest order constraints and periodicity imply

$$e_+ \cdot e_+ = e_- \cdot e_- = 0 \text{ and } e_+ \cdot e_- = -\frac{1}{2}. \quad (2.15)$$

³except for pseudoscalar terms, e.g. a term in D=4 which is of order R^{-2} .

We then have

$$\begin{aligned}
T_{--} = & -\frac{R}{a^2}e_- \cdot \partial_- Y - \frac{1}{2a^2}\partial_- Y \cdot \partial_- Y - \frac{\beta}{R}e_+ \cdot \partial_-^3 Y \\
& - \frac{\beta}{R^2}\partial_+ Y \cdot \partial_-^3 Y - \frac{2\beta}{R^2}e_+ \cdot \partial_-^3 Y(e_+ \cdot \partial_- Y + e_- \cdot \partial_+ Y) \\
& - \frac{2\beta}{R^2}(\partial_+^2 Y \cdot e_- \partial_-^2 Y \cdot e_- + e_+ \cdot \partial_-^2 Y e_+ \cdot \partial_-^2 Y) + O(R^{-3}). \tag{2.16}
\end{aligned}$$

We can also expand the Lagrangian in terms of Y ,

$$\begin{aligned}
\mathcal{L} = & -\frac{R^2}{8\pi a^2} + \frac{1}{4\pi a^2}\partial_+ Y \cdot \partial_- Y + \frac{\beta}{\pi R^2}\partial_-^2 Y \cdot e_+ e_- \cdot \partial_+^2 Y \\
& + \frac{\beta}{\pi R^3}[\partial_+^2 Y \cdot e_- \partial_+ Y \cdot \partial_-^2 Y + \partial_+^2 Y \cdot \partial_- Y e_- \cdot \partial_-^2 Y] \\
& + \frac{4\beta}{\pi R^3}\partial_+^2 Y \cdot e_- e_+ \cdot \partial_+^2 Y[e_+ \cdot \partial_- Y + e_- \cdot \partial_+ Y] + O(R^{-4}). \tag{2.17}
\end{aligned}$$

In fact, after a field redefinition, this can be written as

$$\mathcal{L} = -\frac{R^2}{8\pi a^2} + \frac{1}{4\pi a^2}\partial_+ \hat{Y} \cdot \partial_- \hat{Y} + O(R^{-4}), \tag{2.18}$$

where

$$\begin{aligned}
\hat{Y} = & Y - \frac{\beta a^2}{R^2}[e_- \partial_+ \partial_- Y \cdot e_+ + e_+ \partial_+ \partial_- Y \cdot e_-] \\
& + \frac{2\beta a^2}{R^3}[\partial_- Y e_- \cdot \partial_+^2 Y + \partial_+ Y e_+ \cdot \partial_-^2 Y - \partial_+ Y \partial_+ \partial_- Y \cdot e_- - \partial_- Y \partial_+ \partial_- Y \cdot e_+ \\
& - e_- \partial_- Y \cdot \partial_+^2 Y - e_+ \partial_+ Y \cdot \partial_-^2 Y - 4e_- e_+ \cdot \partial_- \partial_+ Y e_+ \cdot \partial_- Y - 4e_+ e_- \cdot \partial_+ \partial_- Y e_- \cdot \partial_+ Y]. \tag{2.19}
\end{aligned}$$

To see this, one must use partial integrations to rewrite the correction terms in \mathcal{L} so that they are proportional to $\partial_+ \partial_- Y$. This is the variation of the lowest order Lagrangian and so these terms can be removed by field redefinition. This procedure cannot be continued to higher orders as one can see from expanding the Lagrangian to the next order in R^{-1} . Since the Lagrangian is just the free field Lagrangian for the new field, operator products for this field can be evaluated just as in free field theory.

Inverting (2.19) and substituting into (2.16), the energy momentum tensor is found to be

$$\begin{aligned}
T_{--} = & -\frac{R}{a^2}e_- \cdot \partial_- \hat{Y} - \frac{1}{2a^2}\partial_- \hat{Y} \cdot \partial_- \hat{Y} - \frac{\beta}{R}e_+ \cdot \partial_-^3 \hat{Y} \\
& - \frac{2\beta}{R^2}(e_+ \cdot \partial_-^3 \hat{Y} e_+ \cdot \partial_- \hat{Y} - e_+ \cdot \partial_-^2 \hat{Y} e_+ \cdot \partial_-^2 \hat{Y}) + O(R^{-3}). \tag{2.20}
\end{aligned}$$

One can then obtain the TT operator product from the simple form of the $\hat{Y}\hat{Y}$ operator product,

$$T_{--}(\tau^-)T_{--}(0) = \frac{12\beta + D}{2(\tau^-)^4} + \frac{2}{(\tau^-)^2}T_{--}(0) + \frac{1}{\tau^-}\partial_- T_{--}(0) + O(R^{-2}), \tag{2.21}$$

as required for conformal invariance. Note that we have this formula without considering higher order corrections to the action or to the transformation law (since their effect appears at higher order in R^{-1}). This formula implies the Virasoro algebra for the corresponding conserved charges, L_n ,

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{12\beta + D}{12}(m^3 - m)\delta_{m,-n}, \quad (2.22)$$

where the central charge is $12\beta + D$ which we require to take the value 26 for a critical string. As noted in [1] this fixes the value of the parameter β ,

$$\beta = \frac{26 - D}{12}. \quad (2.23)$$

The field $\partial_- \hat{Y}$ obeys $\partial_+ \partial_- \hat{Y} = O(R^{-4})$ and hence has an expansion of the form

$$\partial_- \hat{Y}^\mu = a \sum_{m=-\infty}^{\infty} \hat{\alpha}_m^\mu e^{-im\tau^-} + O(R^{-4}). \quad (2.24)$$

The operators $\hat{\alpha}_m^\mu$ satisfy the standard algebra up to $O(R^{-4})$ since the field \hat{Y} obeys the free field OPE up to this order. It is then simple to express the Virasoro generators in terms of the mode operators, $\hat{\alpha}$. We have, including a possible normal ordering constant,

$$\begin{aligned} L_n = & \frac{R}{a} e_- \cdot \hat{\alpha}_n + \frac{1}{2} \sum_{m=-\infty}^{\infty} : \hat{\alpha}_{n-m} \cdot \hat{\alpha}_m : + \frac{\beta}{2} \delta_{n,0} \\ & - \frac{\beta a n^2}{R} e_+ \cdot \hat{\alpha}_n - \frac{\beta a^2 n^2}{R^2} e_{+\mu} e_{+\nu} \sum_{m=-\infty}^{\infty} \hat{\alpha}_{n-m}^\mu \hat{\alpha}_m^\nu + O(R^{-3}). \end{aligned} \quad (2.25)$$

The $\delta_{n,0}$ term can be calculated using the Virasoro algebra and the standard algebra for the mode operators. This is the crucial formula. It determines the form of the spectrum up to $O(R^{-3})$ as we show below.

The spacetime momentum is given by

$$p^\mu = \frac{R}{2a^2} (e_-^\mu + e_+^\mu) + \frac{1}{2a} (\alpha_0^\mu + \tilde{\alpha}_0^\mu) + O(R^{-4}). \quad (2.26)$$

Imposing the physical condition $L_0 = \tilde{L}_0 = 1$ we find a universal form for the $O(R^{-3})$ correction to the ground state energy,

$$(-p^2)^{\frac{1}{2}} = \frac{R}{2a^2} + \frac{\beta - 2}{R} - \frac{a^2}{R^3} (\beta - 2)^2 + O(R^{-4}), \quad (2.27)$$

with the parameter β fixed to be $(26 - D)/12$. We expect that free parameters will enter the spectrum at higher orders. One can go on and check explicitly that there are still only $(D - 2)$ physical oscillations at the first excited level, a fact guaranteed by the critical value of the central charge.

3. Conclusion

We have shown that the Poincaré invariant effective string model predicts a universal spectrum of fluctuations up to and including the R^{-3} term where R is the string length. This result was obtained without the need to include terms at higher order in the Lagrangian because they can only affect the spectrum at yet higher orders. We have carried out the analysis for a wrapped closed string (or torelon) and have not included possible boundary effects for open strings. It is interesting to note, however, that such boundary effects are not expected to influence the spectrum at the order we are considering [6]. The spectrum obtained is precisely the spectrum of the Nambu-Goto string up to and including the R^{-3} term, with Poincaré invariance being expected to force the two to differ at higher orders.

So far we have found no free parameters in the spectrum and hence the terms obtained at R^{-3} are *universal*. The only assumption that has been made in the derivation is that the effective string model should be Poincaré invariant. It would be very interesting to continue the analysis to higher orders to quantify the number of free parameters appearing there. It is certainly worth remarking that the terms in the Lagrangian required for classical conformal invariance actually appear at $O(R^{-6})$ thus one might expect the universal behaviour in the spectrum to persist even beyond R^{-3} .

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